Decidability in Ramsey theory

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Based on joint work with Steve Jackson and William Mance

https://arxiv.org/abs/2412.14917

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October 15, 2024

Overview

Partition regularity

Definition

Let R be an integral domain, let $S \subseteq R$, let $n, m \in \mathbb{N}$ and $p_1, \cdots, p_m : R[x_1, \cdots, x_n$ be arbitrary. The system of equations

$$\begin{aligned} p_1(x_1, \cdots, x_n) &= 0 \\ &\vdots \\ p_m(x_1, \cdots, x_n) &= 0 \end{aligned} \tag{1}$$

is ℓ-**partition regular (p.r.) over** S if for any partition $S = \bigcup_{i=1}^{\ell} C_i$, there is some $1 \leq i_0 \leq \ell$ for which C_{i_0} contais a solution to the system of equations in (1). The system of equations is **partition regular** if it is ℓ-partition regular for all $\ell \in \mathbb{N}$.

Positive results 1/2

The following systems of equations are partition regular over $\mathbb{N}$.

1) $x + y = z$, Schur 1916 [23]

2) van der Waerden 1927 [26] (arithmetic progressions or A.P.s)

$$\begin{aligned} x_1 - x_2 &= x_2 - x_3 \\ &\vdots \\ x_{n-2} - x_{n-1} &= x_{n-1} - x_n, \text{ or equivalently,} \end{aligned}$$

$$\sum_{i=1}^{n-2} (x_{i+2} - 2x_{i+1} + x_i)^2 = 0.$$

3) Brauer 1928 [5] (A.P.s and their common difference)

$$\begin{aligned} x_1 - x_2 &= x_0 \\ &\vdots \\ x_{n-1} - x_n &= x_0 \end{aligned}$$

Positive results 2/2

4) Rado 1933 [21] classified which finite systems of linear equations are p.r.
5) $x - y = p(z)$ with $p(z) \in z\mathbb{Z}[z]$, Bergelson 1996 [1, page 53]
6) Bergelson, Moreira, and Johnson 2017 [3], for $p_i(x) \in x\mathbb{Z}[x]$

$$\begin{aligned} x_1 - x_2 &= p_1(x_0) \\ &\vdots \\ x_{n-1} - x_n &= p_{n-1}(x_0) \end{aligned}$$

7) $x^2 - y^2 = z$, Moreira 2017 [18]
8) $z = x^y$, Sahasrabudhe 2018 [22]

Negative results

The following systems of equations are not partition regular over $\mathbb{N}$.

1) $2x + 3y = z$, Rado 1933 [21]

2) Rado 1933 [21]

$$\begin{aligned} x + 3y &= z_1 \\ x + 2y &= 2z_2 \end{aligned}$$

3) $x + y = z^2$ (ignoring $2 + 2 = 2^2$), Csikvári, Gyarmati, and Sárközy 2012 [8] (see also [15])

4) $x - 2y = z^2$, Di Nasso and Luperi Baglini 2018 [11]

5) $x^2 - 2y^2 = z$, Di Nasso and Luperi Baglini 2018 [11]

6) $x + y = w^3 z^2$, F. and Magner 2022 [12]

7) $2x + 3y = wz^2$, F. and Magner 2022 [12]

8) F. and Magner 2022 [12]

$$\begin{aligned} x_1 + 17y_1 &= w_1 z_1^{100} \\ 9x_2 + 18y_2 &= w_2 z_2^2 \end{aligned}$$

Open problems

The partition regularity of the following systems of equations over $\mathbb{N}$ is not known.

1) $x^2 + y^2 = z^2$ (**VERY** popular)
2) $a(x^2 - y^2) = bz^2 + dw$ (important, cf. [20])
3) $x^3 + y^3 + z^3 = w^3$ (cf. [7])
4) $x^3 + y^3 + z^3 - 3xyz = w^3$
5) $x^4 + y^4 + z^4 = w^4$ (cf. [7])
6) (**VERY** popular, cf. [18])

$$\begin{aligned} w &= xy \\ z &= x + y \end{aligned}$$

7) $2x - 8y = wz^3$ (cf. [12])
8) (cf. [12])

$$\begin{aligned} 16x_1 + 17y_1 &= w_1 z_1^8 \\ 33x_2 - 17y_2 &= w_2 z_2^8 \end{aligned}$$

First main result (a special case)

Theorem (F., Jackson, Mance, 2024+)

1. *Let us assume that Hilbert's 10th problem over $\mathbb{Q}$ is undecidable. There is no computable condition (computer program) to determine whether or not a given polynomial equation is partition regular over $\mathbb{N}$.*
2. *Suppose that R is the ring of integers of an algebraic function field over a finite field of constants. Then there is no computable condition to determine whether or not a given polynomial equation is partition regular over $R \setminus \{0\}$.*

What is computability and decidability?

Suppose that someone asks you whether or not the equations $x^2 - 5x + 6 = 0$ has a root in $\mathbb{Z}$. We can enumerate the elements of $\mathbb{Z}$, and plug them into the equation one by one until we see that 2 and 3 yield solutions. However, if someone asks you (or maybe a non-mathematician) whether or not the equation $x^2 - 5x + 7 = 0$ has a root in $\mathbb{Z}$, then the previous method will not work, because it will never terminate. Generally speaking, it is not possible to determine whether or not **there exists** an element in an infinite set that satisfies a specific property. We can only create a finite/**computable** procedure to solve such questions in the special cases that the question can be simplified (in a logical sense). In the previous example, the simplification is the quadratic formula. This is a simplification since it lets us avoid checking every member of an infinite set. A problem is **decidable** if there is a computable procedure to solve it.

Hilbert's 10th problem (HTP)

Theorem (Matiyasevič, 1971)

There does not exist a computable procedure for determining whether or not a given polynomial $p \in \mathbb{Z}[x_1, \cdots, x_n]$ *has a root in* $\mathbb{Z}$.

This provides a negative answer to the 10th of the 23 problems of David Hilbert from the 1900 International Congress of Math. See [9] for an exposition of the proof of this result, as well as a discussion of the history.

Open Problem: Does there exist a computable procedure for determining whether or not a given polynomial $p \in \mathbb{Z}[x_1, \cdots, x_n]$ has a root in $\mathbb{Q}$?

The latter problem is referred to as Hilbert's 10th problem over $\mathbb{Q}$. It is generally believed that there does not exist such a computable procedure.

Variations of Hilbert's 10th problem

Given a **computable** integral domain R, we let $HTP(R)$ refer to the following statement:
HTP(R): There does not exist a computable procedure to determine if a given $p \in R[x_1, \cdots, x_n]$ has a root in R.
The statement HTP(R) can be true, or false depending on the integral domain R.

Theorem ([27, 19, 10])

Suppose that $R = \overline{\mathbb{F}_p}$ for some prime p, $R = \mathbb{Z}$, or that $R = R'[t]$ for some integral domain R'.

1. *$HTP(R)$ is true.*
2. *There does not exist a computable procedure for determining whether or not a given polynomial $p \in R[x_1, \cdots, x_n]$ has an integer root $(z_1, \cdots, z_n) \in R^n$ with $z_i \neq z_j$ when $i \neq j$.*

Reducing partition regularity to HTP

Lemma (cf. Krawczyk,Byszewski, 2021 [6])

Let R be an integral domain with field of fractions K. For any $m \in \mathbb{N}$ and any $k_1, \cdots, k_m \in K$, the system of equations

$$\frac{z_{3i-2} - z_{3i-1}}{z_{3i}} = k_i \text{ for all } 1 \leq i \leq m, \tag{2}$$

is partition regular over $\mathbb{R} \setminus \{0\}$.

Corollary

Given an integral domain R, and a polynomial $p \in R[x_1, \cdots, x_n]$, p has a root in K if and only if the equation $p'(x_1, \cdots, x_{3n}) = 0$ with

$$p'(x_1, \cdots, x_{3n}) := p\left(\frac{x_1 - x_2}{x_3}, \cdots, \frac{x_{3n-2} - x_{3n-1}}{x_{3n}}\right)\left(\prod_{i=1}^{n} x_{3i}\right)^{deg(p)}$$

is partition regular over $R \setminus \{0\}$.

Density Ramsey Theory: What is density? 1/2

For $A \subseteq \mathbb{N}$ we denote the **natural upper density** of A by

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1, N]|}{N}. \tag{3}$$

In a countable cancellative commutative semigroup $(S, +)$, a **Følner sequence** $\mathcal{F} = (F_n)_{n=1}^{\infty}$ is a sequence of finite sets s.t.

$$\lim_{N\to\infty} \frac{|(s + F_N)\triangle F_N|}{|F_N|} = 0, \text{ for all } s \in S. \tag{4}$$

Given a Følner sequence $\mathcal{F}$ and a set $A \subseteq S$, the **upper density with respect to** $\mathcal{F}$ is given by

$$\overline{d}_{\mathcal{F}}(A) = \lim_{N\to\infty} \frac{|A \cap F_N|}{|F_N|}. \tag{5}$$

The **upper Banach density** of $A \subseteq S$ is given by

$$d^*(A) = \sup\left\{\overline{d}_{\mathcal{F}}(A) \mid \mathcal{F} \text{ is a Følner sequence}\right\}. \tag{6}$$

Density Ramsey Theory: What is density? 2/2

When $(S,+)=(\mathbb{N},\cdot)$ and p_n denotes the n^{th} prime, an example of a Følner sequence $\mathcal{F}=(F_n)_{n=1}^{\infty}$ is given by

$$F_n=\{p_1^{a_1}p_2^{a_2}\cdots p_n^{a_n}\mid 0\le a_i\le n\ \forall\ 1\le i\le n\} \tag{7}$$

The following alternative characterization of upper Banach density was introduced in [16] for the case of $(\mathbb{Z},+)$, then in more generality in [17] and [2]. We only state a special case here.

Theorem

Let $(S,+)$ be a cancellative commutative semigroup. For $A\subseteq S$,

$$d^*(A)=\sup\Big\{\alpha\ge 0\mid \forall\ F\in\mathscr{P}_f(S)\ \exists\ s\in S \quad s.t.\ |(F+s)\cap A|\ge\alpha|F|\Big\}$$

When R is a countable integral domain, we let d^* denote the upper Banach density in $(R,+)$, and $d^*_\times$ the upper Banach density in $(R\setminus\{0\},\cdot)$.

Szemerédi Theorems

Theorem (Szemerédi's Theorem [24])

If $A \subseteq \mathbb{N}$ satisfies $\overline{d}(A) > 0$ (or $d^*(A) > 0$), then A contains arbitrarily long arithmetic progressions.

Theorem (Furstenberg, Katznelson [14])

If $A \subseteq \mathbb{Z}^d$ satisfies $d^*(A) > 0$, then A contains arbitrarily large d-dimensional cubes.

Theorem (Bergelson, Leibman [4])

If $A \subseteq \mathbb{Z}^d$ satisfies $d^*(A) > 0$, and $p_1, \cdots, p_m : \mathbb{Z}^d \to \mathbb{Z}^d$ are polynomial functions with no constant term, then there exists $a, d \in \mathbb{Z}^d \setminus \{(0, \cdots, 0)\}$ for which $\{a + p_i(d)\}_{i=1}^m \subseteq A$.

See also [25, Corollary 1.6] and [13, Corollary 2.12].

Second main result (a special case)

Theorem (F., Jackson, Mance, 2024+)

1. *Let us assume that Hilbert's 10th problem over* $\mathbb{Q}$ *is undecidable. There is no computable procedure (computer program) to determine whether or not a given polynomial equation has a solution in every set* $A \subseteq \mathbb{N}$ *with* $\overline{d}(A) > 0$*. A similar result holds when* $\overline{d}(A) > 0$ *is replaced by* $d^*(A) > 0$*, or by* $d^*_\times(A) > 0$*.*
2. *Suppose that* R *is the ring of integers of an algebraic function field over a finite field of constants. Then there is no computable procedure to determine whether or not a given polynomial equation has a solution in every* $A \subseteq R$ *with* $d^*(A) > 0$*. A similar result holds when* $d^*(A) > 0$ *is replaced by* $d^*_\times(A) > 0$*.*

Reduction to HTP for density Ramsey theory 1/2

Lemma

Let R be a countably infinite integral domain with field of fractions K. For any $m \in \mathbb{N}$ and any $k_1, \cdots, k_m \in K^\times$ we have the following:

1. *If $A \subseteq R$ is such that $d^*(A) > 0$, then A contains a solution to the system of equations*

$$\frac{z_{4i-3} - z_{4i-2}}{z_{4i-1} - z_{4i}} = k_i \text{ for all } 1 \leq i \leq m. \tag{8}$$

   Furthermore, the solution can be taken such that $z_i \neq z_j$ when $i \neq j$.

2. *If $A \subseteq R \setminus \{0\}$ is such that $d^*_\times(A) > 0$, then A contains a solution $(z_1, \cdots, z_{4m})$ to the system (8), such that $z_i \neq z_j$ for $i \neq j$.*

Reduction to HTP for density Ramsey theory 2/2

Corollary

Let R be a countably infinite integral domain with field of fractions K, and let $p \in R[x_1, \cdots, x_n]$.

1. *p has a root in K if and only if for any $A \subseteq R$ with $d^*(A) > 0$, there exist distinct $z_1, \cdots, z_{4n} \in A$ for which $p'(z_1, \cdots, z_{4n}) = 0$, where*

$$\begin{aligned} & p'(z_1, \cdots, z_{4n}) \\ = & p\left(\frac{z_1 - z_2}{z_3 - z_4}, \cdots, \frac{z_{4n-3} - z_{4n-2}}{z_{4n-1} - z_{4n}}\right)\left(\prod_{i=1}^{n}(z_{4n-1} - z_{4n})\right)^{deg(p)}. \end{aligned}$$

2. *p has a root in K if and only if for any $A \subseteq R \setminus \{0\}$ with $d^*_\times(A) > 0$, there exist distinct $z_1, \cdots, z_{4n} \in A$ for which $p'(z_1, \cdots, z_{4n}) = 0$.*

Future work

Question

Can we prove a version of the corollary on the last slide without the assumption that $z_1, \cdots, z_{4n} \in A$ are distinct?

Question

Given a $\ell \in \mathbb{N}$ and a finite system of linear equations, is there a computable condition to determine whether or not the system is ℓ-partition regular over $\mathbb{Z}$ (or over some integral domain R)?

Question

Given a $\delta \in (0,1)$ and a finite system of linear equations, is there a computable condition to determine whether or not the system has a solution in every set $A \subseteq \mathbb{Z}$ with $d^*(A) > \delta$? How about $d^*_\times(A) > \delta$? What is we replace $\mathbb{Z}$ with an integral domain R?

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