Decidability of the partition regularity of polynomial equations

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Title: Decidability in Ramsey Theory

Abstract: Hilbert's 10th problem asked, roughly speaking, whether or not there exists a computer program that could take as input a polynomial $p \in \mathbb{Z}[x_1, \cdots, x_n]$ and determine whether or not p has an integer root. The work of many mathematicians culminated in 1970 with a negative answer to this question, that such an algorithm does not exist. We are now interested in determining whether or not there exists a computer program that can take as input the polynomial p, and determine either of the following:

(1) Is the equation $p(x_1, \cdots, x_n) = 0$ partition regular over $\mathbb{Z} \setminus \{0\}$?

(2) Given a set $A \subseteq \mathbb{Z}$ with positive upper density, does p have a root in A?

In both cases, we show that such a computer program does not exist, but only after assuming Hilbert 10th problem for $\mathbb{Q}$, which is still an open problem in logic. If we replace our ring $\mathbb{Z}$ with some other rings such as $\overline{\mathbb{F}}_p$, or $R[T]$ with R an integral domain, then we can obtain an analogous result unconditionally.