When normality and dynamical normality coincide for nice classes of Cantor series

Seminarium Zakładu Matematyki Dyskretnej

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Definition

For $b \in \mathbb{N}_{\geq 2}$, a number $x \in [0,1]$ is **normal base-b** if for $\ell \in \mathbb{N}$ and any word $w \in \{0, 1, \cdots, b-1\}^{\ell}$, the word w appears in the base-b expansion of x with the correct frequency. More explicitly, if $x = (0.x_1 x_2 \cdots x_n \cdots)_b$, then

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N \mid w = x_n x_{n+1} \cdots x_{n+\ell-1}\} = b^{-\ell}. \qquad (1)$$

Equivalently, x is **normal base-b** if the sequence $(b^n x)_{n=1}^{\infty}$ is uniformly distributed in $[0,1]$. More explicitly, if for any $0 < a < c < 1$ we have

$$\lim_{N\to\infty} \frac{1}{N} \# \{1 \leq n \leq N \mid b^n x \in (a,c)\} = c - a. \qquad (2)$$

Base-b normality 2/2

We observe that for $x \in [0,1]$ with a base-2 expansion of $x = 0.x_1x_2\cdots x_n\cdots$, we have

$$2^n x \pmod 1 \in \begin{cases} [0,\frac{1}{2}) & \text{iff } x_{n+1} = 0 \\ [\frac{1}{2},1) & \text{iff } x_{n+1} = 1 \\ [0,\frac{1}{4}) & \text{iff } (x_{n+1},x_{n+2}) = (0,0) \\ [\frac{1}{4},\frac{2}{4}) & \text{iff } (x_{n+1},x_{n+2}) = (0,1) \\ [\frac{2}{4},\frac{3}{4}) & \text{iff } (x_{n+1},x_{n+2}) = (1,0) \\ [\frac{3}{4},\frac{4}{4}) & \text{iff } (x_{n+1},x_{n+2}) = (1,1). \end{cases}$$

More generally, if $x = (0.x_1x_2\cdots x_n\cdots)_b$ and $w = (w_1,\cdots,w_\ell) \in \{0,1,\cdots,b-1\}^\ell$, then

$$b^n x \pmod 1 \in [\sum_{j=1}^{\ell} \frac{w_j}{b^j}, \sum_{j=1}^{\ell} \frac{w_j}{b^j} + \frac{1}{b^\ell}) \text{ iff } (x_{n+1},\cdots,x_{n+\ell}) = w.$$

Cantor series

Given a basic sequence (sequence of bases) $Q = (q_n)_{n=1}^{\infty} \in \mathbb{N}_{\geq 2}^{\mathbb{N}}$ and some $x \in [0,1]$, the base Q expansion $x = (0.x_1 x_2 \cdots x_n \cdots)_Q$ with $0 \leq x_i < q_i$ is defined by

$$x = \sum_{n=1}^{\infty} x_i \left(\prod_{j=1}^{n} q_j \right)^{-1} = \frac{x_1}{q_1} + \frac{x_2}{q_1 q_2} + \frac{x_3}{q_1 q_2 q_3} + \cdots \tag{3}$$

Normality for a Cantor series

Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$ and an $x = (0.x_1 x_2 \cdots x_n \cdots)_Q \in [0,1]$, we say that x is **Q-normal** if for any block $b = (b_1, \cdots, b_\ell) \in \mathbb{Z}_{\geq 0}^{\ell}$ satisfying

$$M_B(N) := \sum_{n=1}^{N} \left(\prod_{j=1}^{\ell} \frac{1}{q_{n+j}} \mathbb{1}_{[0,q_{n+j})}(b_j) \right) \xrightarrow[N\to\infty]{} \infty,$$

we have

$$\lim_{N\to\infty} \#\{1 \leq n \leq N \mid x_n x_{n+1} \cdots x_{n+\ell-1} = B\} / M_B(N) = 1.$$

Intuitively, if the block B can appear with a positive frequency, then B appears with the correct frequency.

Dynamical normality for a Cantor series

Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$ and an $x \in [0, 1]$, we say that x is **Q-dynamically normal** if the sequence $(x, q_1 x, q_2 q_1 x, \cdots, q_n q_{n-1} \cdots q_2 q_1 x, \cdots)$ is uniformly distributed. For a general basic sequence Q, the notions of Q-normality and Q-dynamical normality don't need to be the same.

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Dynamically generated basic sequences

Definition

A basic sequence $Q = (q_n)_{n=1}^{\infty}$ is **dynamically generated** if there exists an ergodic measure preserving system $(X, \mathscr{B}, \mu, T)$, a measurable function $f : X \to \mathbb{N}_{\geq 2}$, and a point $y \in X$ which is generic with respect to T and each $\{\mathbb{1}_{f^{-1}(\{n\})}\}_{n\geq 2}$ for which $q_n = f(T^n y)$.

1. If $X = \{1\}$ and T is (necessarily) the identity, then we recover base-b, i.e., $q_n = b$ for all n.
2. If $X = \{0, 1\}$, $Tx = x + 1 \pmod 2$, and $f(i) = b_i$, then we get $q_n = b_0$ if n is even and $q_n = b_1$ if n is odd. (See [1])
3. If $X = [0, 1]$, $Tx = x + \alpha \pmod 1$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $f = 2\mathbb{1}_{[0,\frac{1}{2})} + 3\mathbb{1}_{[\frac{1}{2},1)}$, then $(q_n)_{n=1}^{\infty}$ will be almost periodic sequence of 2s and 3s.
4. We may also consider $f(x) = \left\lfloor \frac{1}{x} \right\rfloor$ in the previous example.

Uniform normality of a Cantor series

Let $Q = (q_n)_{n=1}^{\infty}$ be a dynamically generated basic sequences. For each block $B = (b_1, \cdots, b_\ell) \in \mathbb{N}_{\geq 2}^{\mathbb{N}}$, we have that

$$Q_B(N) := \frac{1}{N} \#\{1 \leq n \leq N \mid (q_n, q_{n+1}, \cdots, q_{n+\ell-1}) = B\} \underset{N \to \infty}{\longrightarrow}$$

exists. $x = (0.x_1 x_2 \cdots x_n \cdots)_Q \in [0,1]$ is Q-**uniformly normal** if for any block of digits $D = (d_1, \cdots, d_\ell)$, and any block of bases $B = (b_1, \cdots, b_\ell)$ with $b_j > d_j$ for all j, we have

$$\lim_{N \to \infty} \frac{1}{Q_B(N)} \#\{1 \leq n \leq N \mid (x_n, \cdots, x_{n+\ell-1}) = D \; \& \; (q_n, \cdots, q_{n+\ell-1}) = B\} = \prod_{j=1}^{\ell} \frac{1}{b_j},$$

provided that $Q_B(N) \to \infty$.

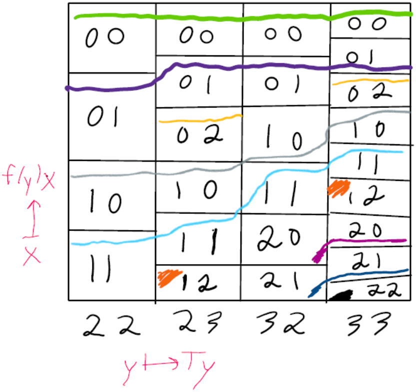
Uniform dynamical normality of a Cantor series

Let $Q = (q_n)_{n=1}^{\infty}$ be a dynamically generated basic sequence, generated by the m.p.s. $(X, \mathscr{B}, \mu, T)$ and the function $f : X \to \mathbb{N}_{\geq 2}$. In particular, we have $q_n = f(T^n y)$ for some $y \in X$. Furthermore, let us assume that this representation is *minimal* in the sense that that σ-algebra generated by f and T is $\mathscr{B}$. A number $x \in [0, 1]$ is Q-**uniformly dynamically normal** if $(S^n(y, x))_{n=1}^{\infty}$ is uniformly distributed in $X \times [0, 1]$, where $S(y, x) = (Ty, f(y)x)$.

A nice equivalence

Theorem

Let $Q = (q_n)_{n=1}^{\infty}$ be a dynamically generated basic sequence. $x \in [0,1]$ is uniformly normal if and only if it is uniformly dynamically normal.



Q-normality means $S^n(y,x)$ visits the colored regions with the correct frequency, where

$S(y,x) = (Ty, f(y)x)$

When Dynamical normality implies normality

Theorem

If $g, k \in \mathbb{N}_{\geq 2}$ and $Q = (q_n)_{n=1}^{\infty} \in \{g, g^2, \cdots, g^k\}^{\mathbb{N}}$ is dynamically generated by a deterministic (zero-entropy) dynamical system, and x is Q-dynamically normal, then x is Q-uniformly dynamically normal. In particular, x is Q-normal. To be more precise, assume that $(X, \mathscr{B}, \mu, T)$ is a deterministic (zero-entropy) dynamical system, and $f : X \to \{g, g^2, \cdots, g^k\}$ (e.g., $\{2, 4\}$) is a measurable function such that $q_n = f(T^n y)$ for some generic point y.

Examples (up to ismorphism) with $(X, \mathscr{B}, \mu) = ([0,1], \mathscr{B}, m)$ and $f = 2\mathbb{1}_{[0,\frac{1}{2})} + 4\mathbb{1}_{[\frac{1}{2},1)}$.

1. $Tx = x + \alpha \pmod 1$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
2. T is any finite interval exchange transformation.
3. T is the Horocycle flow.
4. $(q_n)_{n=1}^{\infty}$ "is" the Thue-Morse sequence with 2s and 4s.

When dynamical normality doesn't imply normality

Sketch: Let $x \in [0, 1]$ be normal base-4 (which is the same as normal base-2). We will now construct a sequence $(q_n))_{n=1}^{\infty} \in \{2, 4\}^{\mathbb{N}}$ in which the 2s always appear in blocks of even size (groups of 2,4,6,...). We let $q_1 = q_2 = 2$ if $x \in [\frac{1}{2}, 1)$ and $q_1 = 4$ otherwise. We now replace x with $4x$ and repeat this procedure inductively to construct the rest of the q_n. The number x is Q-dynamically normal by construction, but it is not Q-normal since the digits 2 and 3 never appear.

When normality doesn't imply dynamical normality

Theorem

There exists a dynamically generated sequences $Q = (q_n)_{n=1}^{\infty}$ and a sequence of digits $(E_n)_{n=1}^{\infty}$ for which $x = E_1 E_2 \cdots E_n \cdots$ is normal but not dynamically normal.

Proof: Let Ω be a probability space and $(q_n(\omega_1))_{n=1}^{\infty}$ a sequences of i.i.d. random variables taking values in $\{2, 3\}$ (can also be done for $\{2,4\}$) with $\mathbb{P}(X_n = 2) = \mathbb{P}(X_n = 3) = \frac{1}{2}$. Consider

$$E_n(\omega_1)(\omega_2) = \begin{cases} 0 & \text{with probability } \frac{1}{2} + \epsilon \text{ if } q_n(\omega_1) = 2 \\ 1 & \text{with probability } \frac{1}{2} - \epsilon \text{ if } q_n(\omega_1) = 2 \\ 0 & \text{with probability } \frac{1}{3} - \epsilon \text{ if } q_n(\omega_1) = 3 \\ 1 & \text{with probability } \frac{1}{3} + \epsilon \text{ if } q_n(\omega_1) = 3 \\ 2 & \text{with probability } \frac{1}{3} \qquad \text{if } q_n(\omega_1) = 3. \end{cases}$$

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When normality implies dynamical normality 1/2

Conjecture

If $g, k \in \mathbb{N}_{\geq 2}$ and $Q = (q_n)_{n=1}^{\infty} \in \{g, g^2, \cdots, g^k\}^{\mathbb{N}}$ is dynamically generated by a deterministic (zero-entropy) system, then every $x \in [0, 1]$ that is Q-normal is also Q-dynamically normal.

Let us now discuss a combinatorial proposition that implies the previous conjecture. For the sake of simplicity we consider $F = \{2, 4\}$. Suppose that $(q_1, q_2, q_3, q_4) = (2, 4, 4, 2)$ and $x = (0.0311.....)_Q$. Then we see that

$$
\begin{aligned}
x &= \frac{0}{2} + \frac{3}{4} \cdot \left(\frac{1}{2}\right) + \frac{1}{4} \cdot \left(\frac{1}{4} \cdot \frac{1}{2}\right) + \frac{1}{2} \cdot \left(\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2}\right) + \dots \\
&= \frac{0}{2} + \frac{1}{2} \cdot 2^{-1} + \frac{1}{2} \cdot 2^{-2} + \frac{0}{2} \cdot 2^{-3} + \frac{1}{2} \cdot 2^{-4} + \frac{1}{2} \cdot 2^{-5} + \dots
\end{aligned}
$$

When normality implies dynamical normality 2/2

In particular, we identify the base 4 with 2 copies of the base 2, and further identify $0_4 = 00_{22}, 1_4 = 01_{22}, 2_4 = 10_{22}$, and $3_4 = 11_{22}$. Under this identification, does a Q-normal number $x \in [0,1]$ become normal base 2? Note that this question could be asked directly since x has not changed, and that the previous discussion is only a combinatorial perspective.

Question

If Q is a dynamically generated basic sequence, and $x \in [0,1]$ is Q-dynamically normal AND Q-normal, must x be Q-uniformly normal?

Mixed bases

Conjecture

If $F \subseteq \mathbb{N}_{\geq 2}$ is finite and $Q = (q_n)_{n=1}^{\infty} \in F^{\mathbb{N}}$ is dynamically generated, then the notions of Q-dynamically normal and Q-normal coincide if and only if Q was generated by a deterministic (zero-entropy) system.

The fundamental problem we are trying to overcome here is that if $F = \{2, 3\}$ instead of $\{2, 4\}$, then we cannot view all of the dynamics as coming from a factor and/or skewproduct of the $\times 2$ map. We genuinely need to consider 2 different transformations, the $\times 2$ map and the $\times 3$ map. Another fundamental problem to overcome is how to allow F to be all of $\mathbb{N}_{\geq 2}$ instead of a finite set. The difficulties here are more technical, so we omit them from here.

References I

[1] D. Airey and B. Mance.
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