Partition regular systems of homogeneous polynomial equations

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Overview

1. A review of Ramsey Theory on semigroups

2. Notions of largeness in semigroups

3. Partition regular systems of polynomial equations

Partition regularity

Definition

Let S be a set, $n, m \in \mathbb{N}$ and $s_0 \in S$ be arbitrary, and $f_1, \cdots, f_m : S^n \to S$ be functions. The system of equations

$$\begin{aligned} f_1(s_1, \cdots, s_n) &= s_0 \\ &\vdots \\ f_m(s_1, \cdots, s_n) &= s_0 \end{aligned} \tag{1}$$

is **partition regular (p.r.)** if for any partition $S = \bigcup_{i=1}^r C_i$, there is some $1 \leq i_0 \leq r$ for which C_{i_0} contais a solution to the system of equations in (1).

Positive results 1/2

The following systems of equations are partition regular over $\mathbb{N}$.

1) $x + y = z$, Schur 1916 [12]

2) van der Waerden 1927 [13] (arithmetic progressions or A.P.s)

$$\begin{aligned} x_1 - x_2 &= x_2 - x_3 \\ &\vdots \\ x_{n-2} - x_{n-1} &= x_{n-1} - x_n \end{aligned}$$

3) Brauer 1928 [3] (A.P.s and their common difference)

$$\begin{aligned} x_1 - x_2 &= x_0 \\ &\vdots \\ x_{n-1} - x_n &= x_0 \end{aligned}$$

Positive results 2/2

4) Rado 1933 [10] classified which finite systems of linear equations are p.r.

5) $x - y = p(z)$ with $p(z) \in z\mathbb{Z}[z]$, Bergelson 1996 [1] (page 53)

6) Bergelson, Moreira, and Johnson 2017 [2], for $p_i(x) \in x\mathbb{Z}[x]$

$$\begin{aligned} x_1 - x_2 &= p_1(x_0) \\ &\vdots \\ x_{n-1} - x_n &= p_{n-1}(x_0) \end{aligned}$$

7) $x^2 - y^2 = z$, Moreira 2017 [8]

8) $z = x^y$, Sahasrabudhe 2018 [11]

Negative results

The following systems of equations are not partition regular over $\mathbb{N}$.

1) $2x + 3y = z$, Rado 1933 [10]

2) Rado 1933 [10]

$$\begin{aligned} x + 3y &= z_1 \\ x + 2y &= 2z_2 \end{aligned}$$

3) $x + y = z^2$ (ignoring $2 + 2 = 2^2$), Csikvári, Gyarmati, and Sárközy 2012 [5]

4) $x - 2y = z^2$, Di Nasso and Luperi Baglini 2018 [6]

5) $x^2 - 2y^2 = z$, Di Nasso and Luperi Baglini 2018 [6]

6) $x + y = w^3 z^2$, F. and Magner 2022 [7]

7) $2x + 3y = wz^2$, F. and Magner 2022 [7]

8) F. and Magner 2022 [7]

$$\begin{aligned} x_1 + 17y_1 &= w_1 z_1^{100} \\ 9x_2 + 18y_2 &= w_2 z_2^2 \end{aligned}$$

Open problems

The partition regularity of the following systems of equations over $\mathbb{N}$ is not known.

1) $x^2 + y^2 = z^2$ (**VERY** popular)
2) $a(x^2 - y^2) = bz^2 + dw$ (important, cf. [9])
3) $x^3 + y^3 + z^3 = w^3$ (cf. [4])
4) $x^3 + y^3 + z^3 - 3xyz = w^3$
5) $x^4 + y^4 + z^4 = w^4$ (cf. [4])
6) (**VERY** popular, cf. [8])

$$\begin{aligned} w &= xy \\ z &= x + y \end{aligned}$$

7) $2x - 8y = wz^3$ (cf. [7])
8) (cf. [7])

$$\begin{aligned} 16x_1 + 17y_1 &= w_1 z_1^8 \\ 33x_2 - 17y_2 &= w_2 z_2^8 \end{aligned}$$

Thick sets and syndetic sets

A **Semigroup** is a pair $(S, \cdot)$ where $\cdot : S \times S \to S$ is an associative operation. For our purposes, we will only focus on the semigroups $(\mathbb{N}, +), (\mathbb{N}, \cdot), (R, +)$, and $(R, \cdot)$, where R is the ring of integers of some number field $K := \mathbb{Q}[\alpha]$. For $s \in S$ and $A \subseteq S$ we define $sA = \{sa \mid a \in A\}$ and $s^{-1}A = \{a \in S \mid sa \in A\}$.

Definition

Let $(S, \cdot)$ be a commutative semigroup and $A \subseteq S$. The set A is **thick** if for any finite set $F \subseteq S$ we have $\bigcap_{f \in F} f^{-1}A \neq \emptyset$. The set A is **syndetic** if there exists a finite set $F \subseteq S$ such that $\bigcup_{f \in F} f^{-1}A = S$.

The set $A_1 = \bigcup_{n=1}^{\infty}[n^2, n^2 + n]$ is a thick set in $(\mathbb{N}, +)$, the set $A_2 = 2\mathbb{N}$ is a syndetic set in $(\mathbb{N}, +)$. For any $(c_n)_{n=1}^{\infty} \subseteq \mathbb{N}$, the set $A_3 = \bigcup_{n=1}^{\infty}\{c_n p_1^{b_1} \cdots p_n^{b_n} \mid 0 \leq b_i \leq n \ \forall\ 1 \leq i \leq n\}$ is thick in $(\mathbb{N}, \cdot)$. The set $A_4 = \{n \in \mathbb{N} \mid v_2(n) \equiv 0 \pmod 2\}$ is syndetic in $(\mathbb{N}, \cdot)$. **Exercise:** The squares are not syndetic in $(\mathbb{N}, \cdot)$.

Piecewise syndetic sets and thickly syndetic sets

Definition

Let $(S, \cdot)$ be a commutative semigroup and $A \subseteq S$. The set A is **piecewise syndetic** if $A = B \cap C$ for some thick set B and some syndetic set C. The set A is **thickly syndetic** if $A \cap A' \neq \emptyset$ for any piecewise syndetic set A'.

Theorem

Let $(S, \cdot)$ be a commutative semigroup and $A \subseteq S$.

(i) *The set A is piecewise syndetic if and only if there exists a finite set $F \subseteq S$ for which $\bigcup_{f \in F} f^{-1}A$ is thick.*

(ii) *The set A is thickly syndetic if and only if for any finite set $F \subseteq S$ the set $\bigcap_{f \in F} f^{-1}A$ is syndetic.*

Exercise: The set A of squarefree numbers is not a piecwise syndetic set in $(\mathbb{N}, +)$, and A^c is a thickly syndetic set.

Ramsey theoretical properties of large sets

Theorem

If $(S, \cdot)$ is a commutative semigroup and $A \subseteq S$ is piecewise syndetic, then for any partition $A = \bigcup_{i=1}^{r} A_i$, at least one of the A_i is piecewise syndetic.

Theorem

Let $(S, \cdot)$ be a commutative semigroup and suppose that **F** *is a translation invariant system of equations. (For example, $x + y = 2z$ over $(\mathbb{N}, +)$ or $xy = z^2$ over $(\mathbb{N}, \cdot)$) The following are equivalent:*

(i) **F** *is partition regular over S.*

(ii) *For any pieceiwse syndetic set $A \subseteq S$,* **F** *has a solution contained in A.*

(iii) *For any very strongly central set A (a special kind of syndetic set)* **F** *has a solution in A.*

Difference of squares generate mult. thick sets

Lemma

Let R be an infinite integral domain and $A \subseteq R$. If A is 'A.P.'-rich (which is implied by additive or multiplicative piecewise syndeticity), then $B := \{x^2 - y^2 \mid x, y \in A\}$ is multiplicatively thick.

Corollary

Let R be an infinite integral domain with field of fractions K. For any $c \in K \setminus \{0\}$, the equation

$$c = \frac{x^2 - y^2}{w^2 - z^2} \tag{2}$$

is partition regular.

First main theorem 1/4

Theorem: Let R be an integral domain with field of fractions K and $n, m \in \mathbb{N}$ arbitrary.

(i) For any $c_0, c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any 'A.P.'-rich set A.

$$\begin{aligned} \frac{c_1 + c_0}{c_1} &= \frac{x^2 - y_1^2}{w^2 - z_1^2} \\ &\vdots \\ \frac{c_m + c_0}{c_m} &= \frac{x^2 - y_m^2}{w^2 - z_m^2} \end{aligned} \tag{3}$$

First main theorem 2/4

(ii) For any $c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} c_1 &= \frac{w}{z}(x_1^2 - y_1^2) \\ &\vdots \\ c_m &= \frac{w}{z}(x_m^2 - y_m^2) \end{aligned} \tag{4}$$

(iii) For any distinct $c_1, c_2 \in \mathbb{Z} \setminus \{0\}$, the system of equations below is not partition regular over $\mathbb{N}$.

$$\begin{aligned} c_1 &= \frac{w_1}{z_1}(x^2 - y^2) \\ c_2 &= \frac{w_2}{z_2}(x^2 - y^2) \end{aligned} \tag{5}$$

First main theorem 3/4

(iv) For any $c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} c_1 &= \frac{x^2 - y_1^2}{4wz_1} \\ &\vdots \\ c_m &= \frac{x^2 - y_m^2}{4wz_m} \end{aligned} \tag{6}$$

(v) For any distinct $c_1, c_2 \in \mathbb{Z} \setminus \{0\}$, the system of equations below is not partition regular over $\mathbb{N}$.

$$\begin{aligned} c_1 &= \frac{x^2 - y^2}{w_1 z_1} \\ c_2 &= \frac{x^2 - y^2}{w_2 z_2} \end{aligned} \tag{7}$$

First main theorem 4/4

(vi) Suppose that $f_1, \cdots, f_m : K^n \to K$ are homogeneous of degree 1. The system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} zf_1(t_1, \cdots, t_n) &= x_1^2 - y_1^2 \\ &\vdots \\ zf_m(t_1, \cdots, t_n) &= x_m^2 - y_m^2 \end{aligned} \tag{8}$$

(vii) And if $f_1, \cdots, f_m : K^n \to K$ are homogeneous of degree 3, then the same is true of the following system of equations:

$$\begin{aligned} f_1(t_1, \cdots, t_n) &= z(x_1^2 - y_1^2) \\ &\vdots \\ f_m(t_1, \cdots, t_n) &= z(x_m^2 - y_m^2) \end{aligned} \tag{9}$$

Example 1/2

Corollary

The following system of equations is partition regular over $\mathbb{Z}$.

$$\begin{aligned} z(2r+3t) &= x_1^2 - y_1^2 \\ z(3r+2t) &= x_2^2 - y_2^2 \\ z\frac{r^2}{t} &= x_3^2 - y_3^2 \\ z\frac{t^2}{r} &= x_4^2 - y_4^2 \\ z\frac{5r^3-7t^3}{2r^2+5t^2} &= x_5^2 - y_5^2 \end{aligned} \tag{10}$$

Example 2/2

Corollary

The following system of equations is partition regular over $\mathbb{Z}$.

$$\begin{aligned} r^3 &= z(x_1^2 - y_1^2) \\ t^3 &= z(x_2^2 - y_2^2) \\ r^3 + t^3 &= z(x_3^2 - y_3^2) \\ 2r^3 - 3r^2t + 7rt^2 - t^3 &= z(x_4^2 - y_4^2) \\ \left\lfloor 2^{\frac{t}{r}} \right\rfloor \left\lfloor \ln\left(\frac{r}{t}\right) \right\rfloor \frac{5r^4 + 7t^4}{9r - 17t} &= z(x_5^2 - y_5^2) \end{aligned} \tag{11}$$

Future work 1/3

Conjecture

Let R be an infinite integral domain and $A \subseteq R$. If A is multiplicatively syndetic, then $B := \{x^2 - y^2 \mid x, y \in A\}$ is multiplicatively thickly syndetic.

Corollary

The following system of equations is (will be) partition regular over $\mathbb{N}$.

$$\begin{aligned} z^3 &= w(x_1^2 - y_1^2) \\ wz &= x_2^2 - y_2^2 \end{aligned} \tag{12}$$

In general, an affirmative answer to Conecture 11 would allow us to combine many of the previous P.R. systems of equations into even bigger P.R. systems of equations.

A cubic form generating mult. thick sets

Lemma

Let R be an infinite integral domain containing a solution ζ to $x^2 + x + 1 = 0$ and $A \subseteq R$. If A is 'A.P.'-rich, then $B := \{x^3 + y^3 + z^3 - 3xyz \mid x, y, z \in A\}$ is multiplicatively thick.

Corollary

Let R be an infinite integral domain containing a solution ζ to $x^2 + x + 1 = 0$ and let K be its field of fractions. For any $c \in K \setminus \{0\}$, the equation

$$c = \frac{x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1}{x_2^3 + y_2^3 + z_2^3 - 3x_2y_2z_2} \tag{13}$$

is partition regular.

$0 = x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \zeta y + \zeta^2 z)(x + \zeta^2 y + \zeta z)$ is nontrivially partition regular over $\mathbb{Z}[\zeta]$ but not $\mathbb{Z}$.

Second main theorem 1/4

Theorem: Let R be an infinite integral domain containing a solution ζ to $x^2+x+1=0$ and let K be its field of fractions.

(i) For any $c_0, c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} \frac{c_1+c_0}{c_1} &= \frac{x^3+y_1^3+z_1^3-3xy_1z_1}{u^3+w_1^3+v_1^3-3yw_1v_1} \\ &\vdots \\ \frac{c_m+c_0}{c_m} &= \frac{x^3+y_m^3+z_m^3-3xy_mz_m}{u^3+w_m^3+v_m^3-3uw_mv_m} \end{aligned} \tag{14}$$

Second main theorem 2/4

(ii) For any $c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} c_1 &= \frac{w}{z}(x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1) \\ &\vdots \\ c_m &= \frac{w}{z}(x_m^3 + y_m^3 + z_m^3 - 3x_my_mz_m) \end{aligned} \tag{15}$$

(iii) For any distinct $c_1, c_2 \in \mathbb{Z} \setminus \{0\}$, the system of equations below is not partition regular over $\mathbb{Z}$.

$$\begin{aligned} c_1 = \frac{w_1}{z_1}(x^3 + y^3 + z^3 - 3xyz) \\ c_2 = \frac{w_2}{z_2}(x^3 + y^3 + z^3 - 3xyz) \end{aligned} \tag{16}$$

Second main theorem 3/4

(iv) For any $c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} c_1 &= \frac{x^3 + y_1^3 + z_1^3 - 3xy_1z_1}{27uvw_1} \\ &\vdots \\ c_m &= \frac{x^3 + y_m^3 + z_m^3 - 3xy_mz_m}{27uvw_m} \end{aligned} \tag{17}$$

(v) For any distinct $c_1, c_2 \in \mathbb{Z} \setminus \{0\}$, the system of equations below is not partition regular over $\mathbb{Z}$.

$$\begin{aligned} c_1 &= \frac{x^3 + y^3 + z^3 - 3xyz}{u_1v_1w_1} \\ c_2 &= \frac{x^3 + y^3 + z^3 - 3xyz}{u_2v_2w_2} \end{aligned} \tag{18}$$

Second main theorem 4/4

(vi) Suppose that $f_1, \cdots, f_m : K^n \to K$ are homogeneous of degree 2. The system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} zf_1(t_1, \cdots, t_n) &= x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1 \\ &\vdots \\ zf_m(t_1, \cdots, t_n) &= x_m^3 + y_m^3 + z_m^3 - 3x_my_mz_m \end{aligned} \tag{19}$$

(vii) And if $f_1, \cdots, f_m : K^n \to K$ are homogeneous of degree 4, then the same is true of the following system of equations:

$$\begin{aligned} f_1(t_1, \cdots, t_n) &= z(x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1) \\ &\vdots \\ f_m(t_1, \cdots, t_n) &= z(x_m^3 + y_m^3 + z_m^3 - 3x_my_mz_m) \end{aligned} \tag{20}$$

Future work 2/3

Conjecture

Let R be an infinite integral domain containing a solution to $x^2 + x + 1 = 0$ and $A \subseteq R$. If A is multiplicatively syndetic, then $B := \{x^3 + y^3 + z^3 - 3xyz \mid x, y, z \in A\}$ is multiplicatively thickly syndetic.

Corollary

The following system of equations is (will be) partition regular over $\mathbb{N}$.

$$\begin{aligned} t^4 &= s(x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1) \\ st^2 &= x_2^3 + y_2^3 + z_2^3 - 3x_2y_2z_2 \end{aligned} \tag{21}$$

In general, an affirmative answer to Conecture 15 would allow us to combine many of the previous P.R. systems of equations into even bigger P.R. systems of equations.

Third main theorem 1/3

Theorem: Let R be an integral domain and let $\mathbf{A} = (a_{i,j})_{1\leq i,j\leq n} \in M_{n\times n}(R)$ satisfy $\det(\mathbf{A}) \neq 0$ and $\sum_{j=1}^{n} a_{1,j} = 0$. Let

$$g_{\mathbf{A}}(x_1, \cdots, x_n) = \prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{i,j}x_i\right). \tag{22}$$

(i) If $A \subseteq R$ if 'A.P.'-rich, then $B := \{g_{\mathbf{A}}(x_1, \cdots, x_n) \mid x_1, \cdots, x_n \in A\}$ is multiplicatively thick.

Third main theorem 2/3

(ii) Suppose that $f_1, \cdots, f_m : R^n \to R$ are homogeneous of degree $n-1$. The system of equations below is partition regular, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} zf_1(t_1, \cdots, t_n) &= g_{\mathbf{A}}(x_{1,1}, \cdots, x_{1,n}) \\ &\vdots \\ zf_m(t_1, \cdots, t_n) &= g_{\mathbf{A}}(x_{m,1}, \cdots, x_{m,n}) \end{aligned} \tag{23}$$

(iii) And if $f_1, \cdots, f_m : R^n \to R$ are homogeneous of degree $n+1$, then the same is true of the following system:

$$\begin{aligned} f_1(t_1, \cdots, t_n) &= zg_{\mathbf{A}}(x_{1,1}, \cdots, x_{1,n}) \\ &\vdots \\ f_m(t_1, \cdots, t_n) &= zg_{\mathbf{A}}(x_{m,1}, \cdots, x_{m,n}) \end{aligned} \tag{24}$$

Third main theorem 3/3

Theorem

Let $\mathbf{A} = (a_{i,j})_{1 \leq i,j \leq n} \in M_{n \times n}(\mathbb{Z} \setminus \{0\})$ *be such that for* $1 \leq i \leq n$ *and* $I \subseteq [1, n]$, *we have* $\sum_{j \in I} a_{i,j} \neq 0$ *unless* $|I| < 2$ *or* $a_{i,j} = 0$ *for some* $j \in I$. *For* $\emptyset \neq I \subseteq [1, n]$ *let* $c_I = \prod_{i=1}^{n} (\sum_{j \in I} a_{i,j})$. *If* $c \in R \setminus \{c_I\}$, *then*

$$ct^{n+1} = zg_{\mathbf{A}}(x_1, \cdots, x_n) \tag{25}$$

is not partition regular over $\mathbb{N}$.

Future work 3/3

Question

(i) *For what* $a, b, c, d \in \mathbb{Z} \setminus \{0\}$ *is the equation*

$$z^3 = w(ax + by)(cx + dy) \tag{26}$$

partition regular over $\mathbb{N}$*?*

(ii) *In the previous theorem is the condition that* $\sum_{j=1}^{n} a_{i,j} = 0$ *a necessary condition? How about the fact that* $\mathbf{A}$ *has nonzero entries? Moreover, if* A *is multiplicatively syndetic, will* B *be multiplicatively thickly syndetic, or will that require additional assumptions of* $\mathbf{A}$*?*

References I

[1] V. Bergelson.
Ergodic Ramsey theory—an update.
In *Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994)*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 1–61. Cambridge Univ. Press, Cambridge, 1996.

[2] V. Bergelson, J. H. Johnson, Jr., and J. Moreira.
New polynomial and multidimensional extensions of classical partition results.
J. Combin. Theory Ser. A, 147:119–154, 2017.

[3] R. Brauer.
Untersuchungen über die arithmetischen Eigenschaften von Gruppen linearer Substitutionen.
Math. Z., 28(1):677–696, 1928.

References II

[4] S. Chow, S. Lindqvist, and S. Prendiville.
Rado's criterion over squares and higher powers.
J. Eur. Math. Soc. (JEMS), 23(6):1925–1997, 2021.

[5] P. Csikvári, K. Gyarmati, and A. Sárközy.
Density and Ramsey type results on algebraic equations with restricted solution sets.
Combinatorica, 32(4):425–449, 2012.

[6] M. Di Nasso and L. Luperi Baglini.
Ramsey properties of nonlinear Diophantine equations.
Adv. Math., 324:84–117, 2018.

[7] S. Farhangi and R. Magner.
On the partition regularity of $ax + by = cw^m z^n$.
Integers, 2022 (to appear).

References III

[8] J. Moreira.
Monochromatic sums and products in $\mathbb{N}$.
Ann. of Math. (2), 185(3):1069–1090, 2017.

[9] S. Prendiville.
Counting monochromatic solutions to diagonal diophantine equations.
Discrete Anal., pages Paper No. 14, 47, 2021.

[10] R. Rado.
Studien zur Kombinatorik.
Math. Z., 36(1):424–470, 1933.

[11] J. Sahasrabudhe.
Exponential patterns in arithmetic Ramsey theory.
Acta Arith., 182(1):13–42, 2018.

References IV

[12] I. Schur.
Uber die kongruenz $x^m + y^m = z^m \pmod{p}$.
Jahresber. Dtsch. Math, 25:114–117, 1916.

[13] B. van der Waerden.
Beweis einer baudetschen vermutung.
Nieuw Arch. Wiskd, 15:212–216, 1927.