

MAJORITY C-COLORING OF GRAPHS

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Inspired by the majority colorings and C-colorings, we introduce and study the majority C-coloring of graphs [1]. In such a vertex coloring, every vertex shares its color with at least half of its neighbors.

Formally, let $G = (V, E)$ be an undirected graph and k a positive integer. A *majority C-coloring* (C_{\geq} -coloring, for short) of G with k colors is a surjective function $\phi : V \rightarrow [k]$ such that every vertex $v \in V$ satisfies

$$|\{u \in N(v) : \phi(u) = \phi(v)\}| \geq \frac{\deg_G(v)}{2}. \quad (1)$$

The largest integer k for which G admits a C_{\geq} -coloring is called *majority C-chromatic number* (C_{\geq} -chromatic number, for short) and it is denoted by $\bar{\chi}_{\geq}(G)$. A majority C-coloring of a graph G with $\bar{\chi}_{\geq}(G)$ colors is called a $\bar{\chi}_{\geq}$ -coloring of G .

For a majority C-coloring ϕ of G that uses k colors, we define the color classes $V_i = \phi^{-1}(i)$, for $i \in [k]$. If a vertex v belongs to the color class V_i , condition (1) can be formulated equivalently as

$$|N(v) \cap V_i| \geq |N(v) \cap (V \setminus V_i)|. \quad (2)$$

An upper bound on $\bar{\chi}_{\geq}(G)$ is proved in terms of the order, minimum, and maximum degree. Its sharpness is demonstrated by several results over different graph classes. In particular,

$$\bar{\chi}_{\geq}(P_n^k) = \bar{\chi}_{\geq}(C_n^k) = \left\lfloor \frac{n}{k+1} \right\rfloor$$

is true for the k -th power of a path and a cycle if $n \geq k+2$. Further,

$$\bar{\chi}_{\geq}(G) = \frac{n-d}{3}$$

holds if G is a (claw, K_4)-free cubic graph and contains d diamonds.

It is also shown that the majority C-chromatic number is not monotone under edge deletion. In fact, both the lower and upper bounds are sharp in the inequality chain

$$\bar{\chi}_{\geq}(G) - 2 \leq \bar{\chi}_{\geq}(G - e) \leq \bar{\chi}_{\geq}(G) + 1.$$

The minimum and maximum number of edges in an n -vertex graph G with $\bar{\chi}_{\geq}(G) = k$ are determined for every n and k . It is also pointed out that the classical chromatic number $\chi(G)$ and $\bar{\chi}_{\geq}(G)$ are incomparable, and the difference $\bar{\chi}_{\geq}(G) - \chi(G)$ can take any positive or negative integer. On the other hand,

$$\bar{\chi}_{\geq}(G) + \chi(G) \leq n + 1$$

holds for every graph G of order n .

The decision problem of whether $\bar{\chi}_{\geq}(G) \geq k$ holds is NP-complete for every fixed $k \geq 2$. In contrast, some sufficient conditions for $\bar{\chi}_{\geq}(G) \geq 2$ are proved, and a linear-time algorithm is presented that determines $\bar{\chi}_{\geq}(T)$ if T is a tree.

References

- [1] C. Bujtás, M. Dettlaff, H. Furmańczyk, A. Laskowska, *Majority C-coloring of graphs* (2026) arXiv:2604.20752v1.