# Decidability in Ramsey theory

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# Overview

# Partition regularity

## Definition

Let $R$ be an integral domain, let $S \subseteq R$, let $n, m \in \mathbb{N}$ and $p_1, \cdots, p_m : R[x_1, \cdots, x_n$ be arbitrary. The system of equations

$$\begin{aligned} p_1(x_1, \cdots, x_n) &= 0 \\ &\vdots \\ p_m(x_1, \cdots, x_n) &= 0 \end{aligned} \tag{1}$$

is $\ell$-**partition regular (p.r.) over** $S$ if for any partition $S = \bigcup_{i=1}^{\ell} C_i$, there is some $1 \leq i_0 \leq \ell$ for which $C_{i_0}$ contais a solution to the system of equations in (1). The system of equations is **partition regular** if it is $\ell$-partition regular for all $\ell \in \mathbb{N}$.

# Positive results 1/2

The following systems of equations are partition regular over $\mathbb{N}$.

1) $x + y = z$, Schur 1916 [23]

2) van der Waerden 1927 [26] (arithmetic progressions or A.P.s)

$$\begin{aligned} x_1 - x_2 &= x_2 - x_3 \\ &\vdots \\ x_{n-2} - x_{n-1} &= x_{n-1} - x_n, \text{ or equivalently,} \end{aligned}$$

$$\sum_{i=1}^{n-2} (x_{i+2} - 2x_{i+1} + x_i)^2 = 0.$$

3) Brauer 1928 [5] (A.P.s and their common difference)

$$\begin{aligned} x_1 - x_2 &= x_0 \\ &\vdots \\ x_{n-1} - x_n &= x_0 \end{aligned}$$

# Positive results 2/2

4) Rado 1933 [21] classified which finite systems of linear equations are p.r.
5) $x - y = p(z)$ with $p(z) \in z\mathbb{Z}[z]$, Bergelson 1996 [1, page 53]
6) Bergelson, Moreira, and Johnson 2017 [3], for $p_i(x) \in x\mathbb{Z}[x]$

$$\begin{aligned} x_1 - x_2 &= p_1(x_0) \\ &\vdots \\ x_{n-1} - x_n &= p_{n-1}(x_0) \end{aligned}$$

7) $x^2 - y^2 = z$, Moreira 2017 [18]
8) $z = x^y$, Sahasrabudhe 2018 [22]

# Negative results

The following systems of equations are not partition regular over $\mathbb{N}$.

1) $2x + 3y = z$, Rado 1933 [21]

2) Rado 1933 [21]

$$\begin{aligned} x + 3y &= z_1 \\ x + 2y &= 2z_2 \end{aligned}$$

3) $x + y = z^2$ (ignoring $2 + 2 = 2^2$), Csikvári, Gyarmati, and Sárközy 2012 [8] (see also [15])

4) $x - 2y = z^2$, Di Nasso and Luperi Baglini 2018 [11]

5) $x^2 - 2y^2 = z$, Di Nasso and Luperi Baglini 2018 [11]

6) $x + y = w^3z^2$, F. and Magner 2022 [12]

7) $2x + 3y = wz^2$, F. and Magner 2022 [12]

8) F. and Magner 2022 [12]

$$\begin{aligned} x_1 + 17y_1 &= w_1z_1^{100} \\ 9x_2 + 18y_2 &= w_2z_2^2 \end{aligned}$$

# Open problems

The partition regularity of the following systems of equations over $\mathbb{N}$ is not known.

1) $x^2 + y^2 = z^2$ (**VERY** popular)
2) $a(x^2 - y^2) = bz^2 + dw$ (important, cf. [20])
3) $x^3 + y^3 + z^3 = w^3$ (cf. [7])
4) $x^3 + y^3 + z^3 - 3xyz = w^3$
5) $x^4 + y^4 + z^4 = w^4$ (cf. [7])
6) (**VERY** popular, cf. [18])

$$\begin{aligned} w &= xy \\ z &= x + y \end{aligned}$$

7) $2x - 8y = wz^3$ (cf. [12])
8) (cf. [12])

$$\begin{aligned} 16x_1 + 17y_1 &= w_1 z_1^8 \\ 33x_2 - 17y_2 &= w_2 z_2^8 \end{aligned}$$

# First main result (a special case)

**Theorem (F., Jackson, Mance, 2024+)**

1. *Let us assume that Hilbert's 10th problem over $\mathbb{Q}$ is undecidable. There is no computable condition (computer program) to determine whether or not a given polynomial equation is partition regular over $\mathbb{N}$.*
2. *Suppose that $R = \overline{\mathbb{F}_p}$ for some prime $p$, or that $R = R'[t]$ where $R'$ is an integral domain. Then there is no computable condition to determine whether or not a given polynomial equation is partition regular over $R \setminus \{0\}$.*

# What is computability and decidability?

Suppose that someone asks you whether or not the equations $x^2 - 5x + 6 = 0$ has a root in $\mathbb{Z}$. We can enumerate the elements of $\mathbb{Z}$, and plug them into the equation one by one until we see that 2 and 3 yield solutions. However, if someone asks you (or maybe a non-mathematician) whether or not the equation $x^2 - 5x + 7 = 0$ has a root in $\mathbb{Z}$, then the previous method will not work, because it will never terminate. Generally speaking, it is not possible to determine whether or not **there exists** an element in an infinite set that satisfies a specific property. We can only create a finite/**computable** procedure to solve such questions in the special cases that the question can be simplified (in a logical sense). In the previous example, the simplification is the quadratic formula. This is a simplification since it lets us avoid checking every member of an infinite set. A problem is **decidable** if there is a computable procedure to solve it.

# Hilbert's 10th problem (HTP)

**Theorem (Matiyasevič, 1971)**

*There does not exist a computable procedure for determining whether or not a given polynomial $p \in \mathbb{Z}[x_1, \cdots, x_n]$ has a root in $\mathbb{Z}$.*

This provides a negative answer to the 10th of the 23 problems of David Hilbert from the 1900 International Congress of Math. See [9] for an exposition of the proof of this result, as well as a discussion of the history.

**Open Problem:** Does there exist a computable procedure for determining whether or not a given polynomial $p \in \mathbb{Z}[x_1, \cdots, x_n]$ has a root in $\mathbb{Q}$?

The latter problem is referred to as Hilbert's 10th problem over $\mathbb{Q}$. It is generally believed that there does not exist such a computable procedure.

# Variations of Hilbert's 10th problem

Given a **computable** integral domain $R$, we let $HTP(R)$ refer to the following statement:

**HTP($R$):** There does not exist a computable procedure to determine if a given $p \in R[x_1, \cdots, x_n]$ has a root in $R$.

The statement HTP($R$) can be true, or false depending on the integral domain $R$.

**Theorem ([27, 19, 10])**

*Suppose that $R = \overline{\mathbb{F}_p}$ for some prime $p$, $R = \mathbb{Z}$, or that $R = R'[t]$ for some integral domain $R'$.*

1. *$HTP(R)$ is true.*
2. *There does not exist a computable procedure for determining whether or not a given polynomial $p \in R[x_1, \cdots, x_n]$ has an integer root $(z_1, \cdots, z_n) \in R^n$ with $z_i \neq z_j$ when $i \neq j$.*

# Reducing partition regularity to HTP

**Lemma (cf. Krawczyk,Byszewski, 2021 [6])**

*Let $R$ be an integral domain with field of fractions $K$. For any $m \in \mathbb{N}$ and any $k_1, \cdots, k_m \in K$, the system of equations*

$$\frac{z_{3i-2} - z_{3i-1}}{z_{3i}} = k_i \text{ for all } 1 \leq i \leq m, \quad (2)$$

*is partition regular over $\mathbb{R} \setminus \{0\}$.*

**Corollary**

*Given an integral domain $R$, and a polynomial $p \in R[x_1, \cdots, x_n]$, $p$ has a root in $K$ if and only if the equation $p'(x_1, \cdots, x_{3n}) = 0$ with*

$$p'(x_1, \cdots, x_{3n}) := p\left(\frac{x_1 - x_2}{x_3}, \cdots, \frac{x_{3n-2} - x_{3n-1}}{x_{3n}}\right)\left(\prod_{i=1}^{n} x_{3i}\right)^{deg(p)}$$

*is partition regular over $R \setminus \{0\}$.*

# Density Ramsey Theory: What is density? 1/2

For $A \subseteq \mathbb{N}$ we denote the **natural upper density** of $A$ by

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1, N]|}{N}. \tag{3}$$

In a countable cancellative commutative semigroup $(S, +)$, a **Følner sequence** $\mathcal{F} = (F_n)_{n=1}^{\infty}$ is a sequence of finite sets s.t.

$$\lim_{N\to\infty} \frac{|(s + F_N)\triangle F_N|}{|F_N|} = 0, \text{ for all } s \in S. \tag{4}$$

Given a Følner sequence $\mathcal{F}$ and a set $A \subseteq S$, the **upper density with respect to** $\mathcal{F}$ is given by

$$\overline{d}_{\mathcal{F}}(A) = \lim_{N\to\infty} \frac{|A \cap F_N|}{|F_N|}. \tag{5}$$

The **upper Banach density** of $A \subseteq S$ is given by

$$d^*(A) = \sup\left\{\overline{d}_{\mathcal{F}}(A) \mid \mathcal{F} \text{ is a Følner sequence}\right\}. \tag{6}$$

# Density Ramsey Theory: What is density? 2/2

When $(S,+) = (\mathbb{N},\cdot)$ and $p_n$ denotes the $n^{\text{th}}$ prime, an example of a Følner sequence $\mathcal{F} = (F_n)_{n=1}^{\infty}$ is given by

$$F_n = \{p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} \mid 0 \leq a_i \leq n \ \forall\ 1 \leq i \leq n\} \tag{7}$$

The following alternative characterization of upper Banach density was introduced in [16] for the case of $(\mathbb{Z},+)$, then in more generality in [17] and [2]. We only state a special case here.

**Theorem**

*Let $(S,+)$ be a cancellative commutative semigroup. For $A \subseteq S$,*

$$d^*(A) = \sup\Big\{\alpha \geq 0 \mid \forall\ F \in \mathscr{P}_f(S)\ \exists\ s \in S \\ s.t.\ |(F+s) \cap A| \geq \alpha|F|\Big\}$$

When $R$ is a countable integral domain, we let $d^*$ denote the upper Banach density in $(R,+)$, and $d^*_\times$ the upper Banach density in $(R\setminus\{0\},\cdot)$.

# Szemerédi Theorems

**Theorem (Szemerédi's Theorem [24])**

*If $A \subseteq \mathbb{N}$ satisfies $\overline{d}(A) > 0$ (or $d^*(A) > 0$), then $A$ contains arbitrarily long arithmetic progressions.*

**Theorem (Furstenberg, Katznelson [14])**

*If $A \subseteq \mathbb{Z}^d$ satisfies $d^*(A) > 0$, then $A$ contains arbitrarily large $d$-dimensional cubes.*

**Theorem (Bergelson, Leibman [4])**

*If $A \subseteq \mathbb{Z}^d$ satisfies $d^*(A) > 0$, and $p_1, \cdots, p_m : \mathbb{Z}^d \to \mathbb{Z}^d$ are polynomial functions with no constant term, then there exists $a, d \in \mathbb{Z}^d \setminus \{(0, \cdots, 0)\}$ for which $\{a + p_i(d)\}_{i=1}^m \subseteq A$.*

See also [25, Corollary 1.6] and [13, Corollary 2.12].

# Second main result (a special case)

**Theorem (F., Jackson, Mance, 2024+)**

1. *Let us assume that Hilbert's 10th problem over $\mathbb{Q}$ is undecidable. There is no computable procedure (computer program) to determine whether or not a given polynomial equation has a solution in every set $A \subseteq \mathbb{N}$ with $\overline{d}(A) > 0$. A similar result holds when $\overline{d}(A) > 0$ is replaced by $d^*(A) > 0$, or by $d^*_\times(A) > 0$.*
2. *Suppose that $R = \overline{\mathbb{F}_p}$ for some prime $p$, or that $R = R'[t]$ where $R'$ is an integral domain. Then there is no computable procedure to determine whether or not a given polynomial equation has a solution in every $A \subseteq R$ with $d^*(A) > 0$. A similar result holds when $d^*(A) > 0$ is replaced by $d^*_\times(A) > 0$.*

# Reduction to HTP for density Ramsey theory 1/2

## Lemma

*Let $R$ be a countably infinite integral domain with field of fractions $K$. For any $m \in \mathbb{N}$ and any $k_1, \cdots, k_m \in K^\times$ we have the following:*

1. *If $A \subseteq R$ is such that $d^*(A) > 0$, then $A$ contains a solution to the system of equations*

$$\frac{z_{4i-3} - z_{4i-2}}{z_{4i-1} - z_{4i}} = k_i \text{ for all } 1 \leq i \leq m. \tag{8}$$

   *Furthermore, the solution can be taken such that $z_i \neq z_j$ when $i \neq j$.*

2. *If $A \subseteq R \setminus \{0\}$ is such that $d^*_\times(A) > 0$, then $A$ contains a solution $(z_1, \cdots, z_{4m})$ to the system (8), such that $z_i \neq z_j$ for $i \neq j$.*

# Reduction to HTP for density Ramsey theory 2/2

## Corollary

*Let $R$ be a countably infinite integral domain with field of fractions $K$, and let $p \in R[x_1, \cdots, x_n]$.*

1. *$p$ has a root in $K$ if and only if for any $A \subseteq R$ with $d^*(A) > 0$, there exist distinct $z_1, \cdots, z_{4n} \in A$ for which $p'(z_1, \cdots, z_{4n}) = 0$, where*

$$\begin{aligned} &p'(z_1, \cdots, z_{4n}) \\ =&p\left(\frac{z_1 - z_2}{z_3 - z_4}, \cdots, \frac{z_{4n-3} - z_{4n-2}}{z_{4n-1} - z_{4n}}\right)\left(\prod_{i=1}^{n}(z_{4n-1} - z_{4n})\right)^{deg(p)}. \end{aligned}$$

2. *$p$ has a root in $K$ if and only if for any $A \subseteq R \setminus \{0\}$ with $d^*_\times(A) > 0$, there exist distinct $z_1, \cdots, z_{4n} \in A$ for which $p'(z_1, \cdots, z_{4n}) = 0$.*

# Future work

**Question**

*Can we prove a version of the corollary on the last slide without the assumption that $z_1, \cdots, z_{4n} \in A$ are distinct?*

**Question**

*Given a $\ell \in \mathbb{N}$ and a finite system of linear equations, is there a computable condition to determine whether or not the system is $\ell$-partition regular over $\mathbb{Z}$ (or over some integral domain $R$)?*

**Question**

*Given a $\delta \in (0, 1)$ and a finite system of linear equations, is there a computable condition to determine whether or not the system has a solution in every set $A \subseteq \mathbb{Z}$ with $d^*(A) > \delta$? How about $d^*_\times(A) > \delta$? What is we replace $\mathbb{Z}$ with an integral domain $R$?*

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